Section 4.7

Applied Optimization

The Closed Interval Method

For a continuous function f on a closed interval $[a, b]$:

(1) Find the values of f at the **critical numbers** in (a, b).

(2) Find the values of f at the **endpoints** of the interval.

(3) The largest (smallest) value is the absolute max (min) value on $[a, b]$.

First Derivative Test

Let c be a critical number of a continuous function f.

(I) If f' changes from + to − at c, then c is a local maximum.

(II) If f' changes from − to + at c, then c is a local minimum.

(III) If f' does not change sign at c, then c is not a local extremum.

Second Derivative Test

Suppose f'' is continuous near c and $f'(c) = 0$.

(I) If $f''(c) < 0$, then c is a local maximum.

(II) If $f''(c) > 0$, then c is a local minimum.

(II) If $f''(c) = 0$, then the test is inconclusive.

Optimization Problems

Solving Optimization Problems

(1) Draw a diagram (if applicable) and fix notation.

(2) What are you attempting to optimize? (the "objective function")
What are the constraints?

(3) What are the constants and variables?
What are the domains for the variables?

(4) Use the constraints to rewrite the objective function in terms of a single variable.

(5) Use calculus to find the absolute minimum or maximum of the objective function on the appropriate domain.

Optimization Problems

Example 1: A piece of wire 10 cm long is bent into a rectangle. What dimensions produce the rectangle with maximum area? Link

Fix notation: Let x and y be the side lengths of the rectangle.

Objective function: Area $A = xy$.

Constraint: Perimeter is $2x + 2y = 10$. Simplify the objective function:

$$\begin{aligned} 2x + 2y &= 10 \\ y &= 5 - x \end{aligned} \qquad \begin{aligned} A &= xy = x(5-x) = 5x - x^2 \\ A'(x) &= 5 - 2x \end{aligned}$$

Critical number: $x = 5/2 = 2.5$.
The domain of x is $[0, 5]$, so we can use the Closed Interval Method:

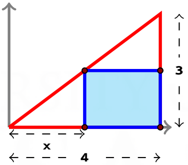
$$\begin{aligned} A(0) &= 0 \\ A(2.5) &= 6.25 \quad - \quad \text{Absolute maximum} \\ A(5) &= 0 \end{aligned}$$

Solution: $x = y = 2.5$ — that is, the rectangle is a **square**.

Example 2: Find two non-negative real numbers x and y such that their sum is 120 and their product is as large as possible.

Example 3: What is the maximum area of a rectangle inscribed in a right triangle with side lengths 3 and 4? The sides of the rectangle are parallel to the legs of the triangle.

Solution: Fix notation: Let (x, y) be the top-left corner of the rectangle.



Link

Objective function: Area, $A = y(4-x)$

Constraint: The hypotenuse of the triangle $y = \frac{3}{4}x$

$$A(x) = \frac{3}{4}x(4-x) \qquad \text{Domain: } [0,4]$$

Critical Numbers in [0,4]: $x = 2$

Closed Interval Method:

$$A(0) = 0$$
$$A(2) = 3 \quad - \quad \text{Absolute maximum}$$
$$A(4) = 0$$

Solution: The maximum area is 3 units^2 with dimensions $2 \times \frac{3}{2}$

Example 4: A box with a square base and open top must have a volume of $32,000\,cm^3$. Find the dimensions of the box that minimize the amount of material used.

Example 5: An oil refinery is located on the north bank of a straight river that is 2 *km* wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 *km* east of the refinery.

The cost of laying pipe is \$400,000 per *km* over land to a point P on the north bank and \$800,000 per *km* under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

Example 6: A landscape architect wishes to enclose a rectangular garden of area $1000\,m^2$. On one side will be a brick wall costing \$90 per linear meter and on the other three sides will be a metal fence costing \$30 per linear meter. What dimensions minimize the total cost?

Example 7: A poster of area $6000\,cm^2$ has blank margins of width $10\,cm$ on the top and bottom and $6\,cm$ on the sides. Find the dimensions of the poster that maximize the printed area.

Example 8: Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 1.

Example 9: Boat A leaves a dock at 1 PM and travels due north at a speed of $20\,km/h$. Boat B has been heading due west at $15\,km/h$ and reaches the same dock at 2 PM. How many minutes after 1 PM were the two boats closest together?

Example 10: A rain gutter is to be constructed from a sheet of metal that is 6 *in* wide by bending the 2 *in* on each end upwards at an angle θ. Find the angle θ which will maximize the amount of water able to flow through the gutter.